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Game Semantics for Concurrent and Distributed Systems

Marina Lenisa
(Università di Udine, Italy)

joint work with

Furio Honsell   Daniel Pellarini
(Università di Udine, Italy)

Rekha Redamalla
(Università di Hyderabad, India)

Games

- A wide gamut of interactions and other dynamic phenomena are described using game-based metaphors
- Various notions of games arise in the literature, *e.g.* in semantics, model checking, economics ...
- Games of perfect/non-perfect information, sequential/concurrent games, ...
- Many concepts, no standard definitions: move, position, play, turn, winning condition, payoff function, strategy, ...
- A unifying framework is called for ...

Coalgebraic methods

Coalgebraic methods provide a convenient conceptual setting for describing games:

games	are defined as	elements of a final coalgebra
game operations	are defined as	final morphisms

Part I: the general coalgebraic framework

- We provide a unifying general notion of coalgebraic game subsuming various notions of (possibly non-wellfounded) games, e.g. Conway games, loopy games, and games arising in Game Semantics.
- We define various game operations coalgebraically, *e.g.* Conway disjoint sum, Conway selective sum, negation, and linear logic connectives of Game Semantics.

F. Honsell, M. Lenisa,“Conway Games, algebraically and coalgebraically”, Logical Methods in Computer Science, 7(3), 2011.
F. Honsell, M. Lenisa, R. Redamalla, “Equivalences and Congruences on Infinite Conway Games”, RAIRO Theoretical Informatics and Applications, 46(2),2012.

Part II: categories of coalgebraic games

- We introduce constructions of symmetric monoidal closed categories of non-wellfounded games, which generalize categories of games for semantics as well as Joyal's category of Conway games.
- Disjoint sum provides a category of sequential games in the style of traditional game semantics à la [Abramsky-Jagadeesan-Malacaria00].
- Selective sum provides a new category of concurrent games, halfway between traditional game semantics and concurrent games of [Abramsky-Mellies99, Mellies07, Winskel et al.11-12].

F. Honsell, M. Lenisa, R. Redamalla, "Categories of Coalgebraic Games", MFCS'2012, LNCS.
F. Honsell, M. Lenisa, D. Pellarini, "Categories of Coalgebraic Games with Selective Sum", submitted.

2-player games of perfect information

- 2-players games, Left (L) and Right (R)
- games have positions
- a game is identified with its initial position
- at any position, there are moves for L and R, taking to new positions of the game
- perfect knowledge: all positions are public to both players

Disjoint vs selective sum

- Sum is used to (de)compose games, [Conway01, Ch.14 “How to play several games at once in dozen of different way”].
- In the disjoint sum of two games, at each step, the player who moves chooses one component and performs a move in that component, leaving unchanged the other.
- In the selective sum, at each step, the player who moves can choose to move in one or in both components.
- Disjoint sum reflects an interleaving semantics, while selective sum accommodates a form of (true) parallelism, by allowing the current player (or a team of players) to move in different parts of the board simultaneously.

From sequential to concurrent games

- Categories based on disjoint sum correspond to traditional sequential game models.
- Categories based on selective sum are situated halfway between traditional sequential models and concurrent game models of [Abramsky-Mellies99, Mellies07, Winskel et al.11-12].

Sequential games	Games with selective sum	Concurrent games
global polarization	global polarization	no global polarization
single move	multiple moves	multiple moves

Global polarization: at each step, a single player moves (players move in alternation)

Some technical difficulties

- Defining good semantic models, in particular strategy composition, is not easy, when strategies are allowed to play several moves in a row, cfr. Blass games.
- Working with non-wellfounded (*i.e.* infinite) games and non-losing strategies is problematic, even with disjunctive sum. This is why in game semantics only fixed games are normally considered, *i.e.* no draws are admitted.
- Determinacy for non-wellfounded games can be problematic, cfr [Winskel et al. 2012].

Part I: the general coalgebraic framework

Coalgebraic games

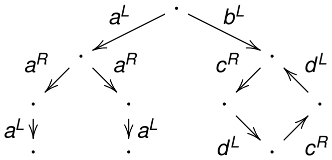
Definition

- Let $\mathcal{A}$ be a set of atoms with functions:
  - $\mu : \mathcal{A} \to \mathcal{M}$ yielding the name of the move,
  - $\lambda : \mathcal{A} \to \{L, R\}$ yielding the player who has moved.
- Let Set^* be the category of sets of a universe satisfying the Antifoundation Axiom.
- Let $F_{\mathcal{A}} : \mathrm{Set}^* \to \mathrm{Set}^*$ be the functor defined by $F_{\mathcal{A}}(X) = \mathcal{P}_{<\kappa}(\mathcal{A} \times X)$.
- Let $(\mathcal{G}_{\mathcal{A}}, id)$ be the final $F_{\mathcal{A}}$-coalgebra.
- A coalgebraic game is a set x in the carrier $\mathcal{G}_{\mathcal{A}}$ of the final coalgebra.

Coalgebraic games are parametric w.r.t. the set of atoms $\mathcal{A}$.

Coalgebraic/set-theoretic games vs games as graphs

- Often games are represented as graphs, where nodes are positions and arcs are moves.
- Coalgebraic games are the minimal graphs up-to bisimilarity.
- Our coalgebraic approach is motivated by the fact that the existence of winning/non-losing strategies (and other equivalences on games) is preserved under graph bisimilarity of games.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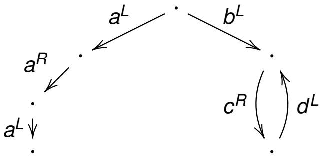
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Strategies

We focus on strategies that always provide an answer, if any, of the player to the moves of the opponent player.

Definition

A strategy σ for LII (*i.e.* L acting as player II) is a partial function $\sigma : FPlay_x^{LII} \to \mathcal{A} \times \mathcal{G}_{\mathcal{A}}$ such that, for any $s \in FPlay_x^{LII}$,

– $\sigma(s) = \langle a, x' \rangle \implies a = L \wedge s\langle a, x' \rangle \in FPlay_x$

– $\exists \langle a, x' \rangle. \, (s\langle a, x' \rangle \in FPlay_x \wedge a = L) \implies s \in dom(\sigma)$.

Similarly, one can define strategies for players LI, RI, RII.

Winning/non-losing strategies

- We endow coalgebraic games x with a payoff function $\nu : Play_x \to \{0, -1, 1\}$.
- A finite play is winning for the player who performs the last move.
- Infinite plays s can be taken to be winning for L/R ($\nu(s) = 1/-1$) or draws ($\nu(s) = 0$).

Definition

A strategy is winning/non-losing for a player, if it generates winning/non-losing plays against any possible counterstrategy.

- Mixed games: the whole class of games, where plays can be winning or draws.
- Fixed games: the subclass of games where all plays are winning for one of the players.

Special instances of the coalgebraic framework

- Conway games and winning strategies are recovered by forgetting about move names and restricting to well-founded games.
- Loopy games of [Berlekamp-Conway-Guy82] and non-losing strategies are obtained, up-to bisimilarity, by forgetting about move names.
- Games of traditional game semantics are fixed alternating coalgebraic games, up-to bisimilarity.
- Mild generalizations of the set of moves capture games of model checking, games in Economics, . . .

Game operations, coalgebraically

- Game operations are defined via final morphisms.
- Disjoint sum: on fixed and mixed games. The two sums have the same coalgebraic structure, but differ by the payoff on infinite plays.
- Selective sum: on fixed and mixed games.
- Negation.
- Linear implication.
- Exponential.
- …

Disjoint sum

Coalgebraic structure. On $x + y$, at each step, the current player selects one of the components and makes a legal move in that component, the other remaining unchanged:

Definition

$$x + y = \{\langle a, x' + y \rangle \mid \langle a, x' \rangle \in x\} \cup \{\langle a, x + y' \rangle \mid \langle a, y' \rangle \in y\}$$

$+ : (\mathcal{G}_{\mathcal{A}} \times \mathcal{G}_{\mathcal{A}}, \alpha_+) \longrightarrow (\mathcal{G}_{\mathcal{A}}, \mathrm{id})$ is a final morphism, for $\alpha_+ : \mathcal{G}_{\mathcal{A}} \times \mathcal{G}_{\mathcal{A}} \longrightarrow F_{\mathcal{A}}(\mathcal{G}_{\mathcal{A}} \times \mathcal{G}_{\mathcal{A}})$ the coalgebra morphism: $\alpha_+(x, y) = \{\langle a, \langle x', y \rangle \rangle \mid \langle a, x' \rangle \in x\} \cup \{\langle a, \langle x, y' \rangle \rangle \mid \langle a, y' \rangle \in y\}$.

Payoff. Two sums arise from the above coalgebraic definition:

- Mixed disjoint sum $+_m$. An infinite play is winning for L (R) if all infinite subplays in the components are winning for L (R), it is a draw otherwise. Conway's disjoint sum.
- Fixed disjoint sum $+_f$. An infinite play is winning for L iff all infinite subplays are winning for L, it is winning for R otherwise. Tensor product of Game Semantics.

Selective sum

Coalgebraic structure. On $x \vee y$, at each step, the next player selects either one (non-ended) or both component games, and makes a legal move in each of the selected components:

Definition

$$x \vee y = \{\langle a, x' \vee y\rangle \mid \langle a, x'\rangle \in x\} \cup \{\langle a, x \vee y'\rangle \mid \langle a, y'\rangle \in y\} \cup \{\langle a, x' \vee y'\rangle \mid \langle a, x'\rangle \in x \ \&\ \langle a, y'\rangle \in y\}.$$

Payoff. Two sums arise from the above coalgebraic definition:

- Mixed selective sum $\vee_m$. An infinite play is winning for L (R) if all infinite subplays in the components are winning for L (R), it is a draw otherwise.
- Fixed selective sum $\vee_f$. An infinite play is winning for L iff all infinite subplays in the components are winning for L, it is winning for R otherwise.

Part II: categories of coalgebraic games

Game category spectrum: disjoint sum as tensor

Objects	Morphisms	Properties
well-founded games	winning strategies	compact closed (Joyal's category)
fixed games	winning strategies	$*$-autonomous (sequential game category of [Abr96])
pairs of fixed games (mixed games $\langle x^-, x^+ \rangle$)	pairs of winning strategies	$*$-autonomous (extends Joyal'category to loopy games)

Game category spectrum: selective sum as tensor

Objects	Morphisms	Properties
well-founded alternating games	winning strategies	SMCC
fixed alternating games	winning strategies	SMCC + comonad
pairs of fixed alternating games (mixed games $\langle x^-, x^+\rangle$)	pairs of winning strategies	SMCC + comonad

Conclusions and future work

- A new general notion of coalgebraic game
  - providing a unified treatment of games
  - subsuming (non-wellfounded) Conway games and Game Semantics
  - shedding light on the relations between them.
- Categories of games and strategies based on disjoint and selective sum.
- New parallel game models situated halfway between traditional sequential and recent concurrent game models of [Abramsky-Mellies99, Mellies06, Winskel et al. 11-12].

? Investigating usefulness of the selective model for modeling distributed systems.

? Extending the selective model by abandoning global polarization.

? Generalizing the coalgebraic framework: probabilistic, stochastic games, imperfect information games, . . .