ULTraS *at Work: Compositionality Metaresults for Bisimulation and Trace Semantics*

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Objectives of Behavioral Metamodels

- *Unifying theories*: offering a uniform view of behavioral models that have already appeared in the literature.
- *Reuse facilities*: providing methodologies, results, and tools that can be instantiated to a wide range of specific behavioral models.
- Support to the development of *new* theories, models, or calculi accounting for certain behavioral aspects.
- This should *reduce* the effort needed for:
    - defining syntax and semantics;
    - setting up meaningful equivalences;
    - studying compositionality properties;
    - investigating equational and logical characterizations;
    - devising verification algorithms;
    - ...

Towards Behavioral Metamodels

- Operational semantic rule formats.
- Segala probabilistic automata / Markov automata.
- Weighted automata.
- ...
- They were *not* developed with the *explicit* purpose of paving the way to unifying theories and reuse facilities.
- More a matter of *ensuring* certain properties or achieving a certain *expressivity* rather than metamodeling.
- However, *not too abstract* as the categorical representations based on coalgebras and bialgebras, hence easier to use.

Some Proposals

- The WLTS metamodel by Klin (no internal nondeterminism):
  - relies on *commutative monoids* to express and combine weights attached to transition labels under a *weight determinacy condition*;
  - equipped with a notion of *weighted bisimilarity* and a *rule format* ensuring the compositionality of bisimulation semantics.
- The FuTS metamodel by De Nicola, Massink, Latella & Loreti.
  - based on *commutative semirings* for a compositional and compact definition of operational semantic rules;
  - supports a precise understanding of similarities and differences among process calculi of the same class.
- The ULTraS metamodel by Bernardo, De Nicola & Loreti.
  - based on *preordered sets equipped with minimum*;
  - focusses on models and equivalences.

Definition of the ULTRAS Metamodel

- $(D, \sqsubseteq_D, \bot_D)$ is a preordered set equipped with minimum $\bot_D$, where each value represents a degree *of one-step reachability* and $\bot_D$ denotes *unreachability*.
- A uniform labeled transition system, ULTRAS, on $(D, \sqsubseteq_D, \bot_D)$ is a triple $\mathcal{U} = (S, A, \longrightarrow)$ where:
  - S is a countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times (S \to D)$ is a transition relation.
- Given a transition $s \xrightarrow{a} \Delta$, function Δ represents the *distribution of reachability* over all possible states from s via that a-transition.
- Set of *reachable states*: $supp(\Delta) \triangleq \{s \in S \mid \Delta(s) \neq \bot_D\} \neq \emptyset$.

Generality of the ULTRAS Metamodel

- The ULTRAS metamodel is general enough to encompass:
  - purely nondeterministic systems: LTS;
  - probabilistic systems: ADTMC, MDP, PA;
  - stochastically timed systems: ACTMC, CTMDP, MA;
  - deterministically timed systems: TA, PTA.
- Bisimulation, testing, and trace metaequivalences on ULTRAS capture most behavioral equivalences on specific models ...
- ... but *not* all of them!
- New equivalences arise when instantiating ULTRAS to models with probabilities and internal nondetermism, while the widely accepted ones are left out.

Summary of Results

- Introduction of *resolutions* in the ULTraS setting.
- Elicitation of a semiring structure within ULTraS, so to reconcile ULTraS and FuTS.
- Redefinition of bisimulation and trace metaequivalences by distinguishing between *pre*- and *post*-metaequivalences, so to capture the equivalences that were left out.
- Congruence metaresults for bisimulation and trace semantics, which confirm a "duality" with respect to parallel composition that shows up in the presence of internal nondeterminism:
  - for bisim., only the post-metaequivalence is always compositional;
  - for trace, only the pre-metaequivalence is always compositional.

Resolutions

- A resolution of a state s of an ULTRAS $\mathcal{U}$ is the result of a possible way of resolving choices starting from s.
- As if a *deterministic scheduler* were applied that, at each step, selects one of the outgoing transitions or no transitions at all.
- A *deterministic* ULTRAS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$.
- Formalization via *locally injective* functions inspired by testing theories for probabilistic and nondeterministic processes.
- $Res(s)$ is the set of resolutions of s.
- $k\text{-}Res(s)$ is the set of k-resolutions of s (for bisimulation semantics).

Reachability-Consistent Semirings

- ULTRAS metaequivalences are parameterized with respect to a *measure function* expressing the degree of *multi-step reachability*.
- Return *sets* of values in the presence of internal nondetermism.
- Always return *single* values when applied to resolutions.
- A semiring $(D, \oplus, \otimes, 0_D, 1_D)$ emerges where:
  - $\oplus$ aggregates values of multi-step reachability along different trajectories starting from the same state;
  - $\otimes$ calculates multi-step reachability from values of consecutive single-step reachability along the same trajectory.
- Consistent with the intuition behind reachability:
  - $0_D = \bot_D$;
  - $d_1 \otimes d_2 \neq 0_D$ whenever $d_1 \neq 0_D \neq d_2$;
  - the sum of arbitrarily many values 1_D is always different from 0_D (known as *characteristic zero*).

From Measure Functions to Measure Schemata

- A measure schema $\mathcal{M}$ for an ULTRAS $\mathcal{U} = (S, A, \longrightarrow)$ over the reachability-consistent semiring $(D, \oplus, \otimes, 0_D, 1_D)$ is a set of *measure functions* $\mathcal{M}_{\mathcal{Z}} : Z \times A^* \times 2^Z \to D$ for each $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in \mathit{Res}(s)$ and $s \in S$, such that:

$$\mathcal{M}_{\mathcal{Z}}(z, \alpha, Z') = \begin{cases} \bigoplus_{z' \in supp(\Delta)} (\Delta(z') \otimes \mathcal{M}_{\mathcal{Z}}(z', \alpha', Z')) & \text{if } \alpha = a\,\alpha' \text{ and } z \xrightarrow{a}_{\mathcal{Z}} \Delta \\ 1_D & \text{if } \alpha = \varepsilon \text{ and } z \in Z' \\ 0_D & \text{otherwise} \end{cases}$$

- The definition applies to $\mathcal{Z} \in k\text{-}\mathit{Res}(s)$ by restricting to traces $\alpha \in A^*$ such that $|\alpha| \leq k$.
- $\mathcal{M}_{\mathrm{nd}}$ for $(\mathbb{B}, \vee, \wedge)$.
- $\mathcal{M}_{\mathrm{pb}}$ for $(\mathbb{R}_{\geq 0}, +, \times)$.

Bisimulation Metaequivalences

- $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ and $\sim^{\text{post}}_{\text{B},\mathcal{M}}$.
- An equivalence relation $\mathcal{B}$ over S is an $\mathcal{M}$-pre-bisimulation iff for all $(s_1, s_2) \in \mathcal{B}$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s_2)$ such that:

$$\mathcal{M}(z_{s_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}(z_{s_2}, a, \bigcup \mathcal{G})$$

- An equivalence relation $\mathcal{B}$ over S is an $\mathcal{M}$-post-bisimulation iff for all $(s_1, s_2) \in \mathcal{B}$ and $a \in A$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s_2)$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}(z_{s_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}(z_{s_2}, a, \bigcup \mathcal{G})$$

- *Groups* of equivalence classes instead of just equivalence classes avoid getting a pre-metaequivalence that is too coarse and ensure transitivity of bisimilarity over continuous state spaces.

Trace Metaequivalences

- $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_2$ iff for all $\alpha \in A^*$ it holds that
  for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$
  (for each $\mathcal{Z}_2 \in Res(s_2)$ there exists $\mathcal{Z}_1 \in Res(s_1)$)
  such that:
  $$\mathcal{M}(z_{s_1}, \alpha, Z_1) \; = \; \mathcal{M}(z_{s_2}, \alpha, Z_2)$$
- $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}} s_2$ iff
  for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$
  (for each $\mathcal{Z}_2 \in Res(s_2)$ there exists $\mathcal{Z}_1 \in Res(s_1)$)
  such that for all $\alpha \in A^*$:
  $$\mathcal{M}(z_{s_1}, \alpha, Z_1) \; = \; \mathcal{M}(z_{s_2}, \alpha, Z_2)$$

Backward Compatibility

- $\sim^{\mathrm{pre}}_{\mathrm{B},\mathcal{M}}$ coincides with the original bisimulation metaequivalence for ULTRAS.
- $\sim^{\mathrm{post}}_{\mathrm{B},\mathcal{M}_{\mathrm{pb}}}$ coincides with the strong bisimulation equivalence of Segala & Lynch (not initially captured).
- $\sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}}$ coincides with the original trace metaequivalence for ULTRAS.
- $\sim^{\mathrm{post}}_{\mathrm{B},\mathcal{M}_{\mathrm{pb}}}$ coincides with the trace-distribution equivalence of Segala (not initially captured).

Discriminating Power

- $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ is finer than $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$.
- $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ and $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ coincide if there is no internal nondeterminism.
- $\sim^{\text{post}}_{\text{T},\mathcal{M}}$ is finer than $\sim^{\text{pre}}_{\text{T},\mathcal{M}}$.
- $\sim^{\text{post}}_{\text{T},\mathcal{M}}$ and $\sim^{\text{pre}}_{\text{T},\mathcal{M}}$ coincide if there is no internal nondeterminism.
- $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ is finer than $\sim^{\text{post}}_{\text{T},\mathcal{M}}$.
- $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ and $\sim^{\text{pre}}_{\text{T},\mathcal{M}}$ are incomparable if there is internal nondet.

Compositionality of Bisimulation Metaequivalences

- Both are congruences with respect to *generalized* operators of action prefix, guarded choice, and nondeterministic choice.
- $\sim^{\rm post}_{\rm B,\mathcal{M}}$ is a congruence with respect to parallel composition, hence so is $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ in the absence of internal nondeterminism.
- In the only reachability-consistent semiring with $|D| = 2$, which is $(\mathbb{B}, \vee, \wedge)$, parallel composition cannot generate values different from true and false.
- $\sim^{\rm pre}_{\rm B,\mathcal{M}_{\rm nd}}$ is a congruence with respect to parallel composition.
- $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ is *not* a congruence with respect to parallel composition when $|D| > 2$ and there is internal nondeterminism.
- $\sim^{\rm post}_{\rm B,\mathcal{M}}$ is the *coarsest* congruence contained in $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ w.r.t. parallel composition in the case of a reachability-consistent *field*, image finiteness, and finitely supported reachability distributions.

Compositionality of Trace Metaequivalences

- Both are congruences with respect to *generalized* operators of action prefix, guarded choice, and nondeterministic choice.
- $\sim^{\text{pre}}_{\text{T},\mathcal{M}}$ is a congruence with respect to parallel composition, hence so is $\sim^{\text{post}}_{\text{T},\mathcal{M}}$ in the absence of internal nondeterminism.
- Based on an alternative characterization of $\sim^{\text{pre}}_{\text{T},\mathcal{M}}$ associating with each state the *set of traces* it can perform in the various resolutions together with their *degree of executability*.
- $\sim^{\text{post}}_{\text{T},\mathcal{M}}$ is *not* a congruence with respect to parallel composition when there is internal nondeterminism.
- Even in the case that $|D| = 2$.
- A coarsest congruence result is impossible.

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Future Work

- Keep putting ULTraS at work.
- Equational characterizations of bisimulation and trace metaequivalences.
- Logical characterizations of bisimulation and trace metaequivalences.
- ...